JOURNAL OF PROPULSION AND POWER
Vol. 16, No. 3, May–June 2000

Adiabatic Interaction of Particles with a Compressible Flow: Basic Mechanics

Francisco Ruiz*
Illinois Institute of Technology, Chicago, Illinois 60616

The final state of the interaction of a compressible flow with a spray or solid particle dispersion in a constant-area straight duct is analyzed using fluid mechanics in a control-volume approach. The variations of the flow Mach number at low values of the liquid volume fraction are calculated for adiabatic interaction between the gas flow and the particles. The flow can be choked by effect of the particles in a way analogous to that of heat addition or wall friction, although isentropic interactions leading to a change from subsonic to supersonic flow are also possible. The final flow condition depends on the initial velocity of the particles and the particle flow rate. The effects of downstream pressure changes are discussed. The interaction can also be represented in a temperature-entropy diagram, yielding lines similar to those obtained in cases with heat addition or wall friction.

Nomenclature

A = local section area
Cp = specific heat at constant pressure
c = speed of sound
k = adiabatic exponent, Cp/Cv
M = Mach number
m_L = mass flow rate of the spray
ms = relative spray/airflow ratio
P = pressure
R_g = ideal gas constant
S = gas specific entropy
T = gas temperature
V = velocity
ρ = density
τ = temperature ratio T_2/T_1

Subscripts

g = of the gas
L = of the particles
1 = before phase interaction
2 = after complete phase interaction
$'$ = conjugated (across a normal shock wave)

Introduction

THE addition of a solid or liquid phase to a gas can lower the speed of sound of the resulting two-phase mixture to very low values.[1] This phenomenon has commonly been used in spray cans to generate underexpanded propellant-liquid jets that diverge on contact with the lower ambient pressure and cause the liquid phase to be finely atomized.[2] Critical flow effects are also encountered in flash-boiling systems, where the separation of vapor bubbles from a liquid flow can lead to choking of the flow.[1] In most cases analyzed in the literature, the liquid volume fraction is significantly larger than that of the gas.

In the particular case of ramjet engines, a liquid fuel is injected into a high subsonic or supersonic flow. In this case, unlike in the other cases just mentioned, the volume occupied by the liquid is negligible (although its mass is not) compared to that of the gas phase. This paper presents a simple analysis for the limit where the particle volume fraction is much smaller than that of the gas. This is the final state encountered in spray systems, fuel injection, and combusting flows. We will see how particles interacting with a high-speed flow can change its Mach number and open the possibility for some interesting applications.

Dispersion of a Spray in a Compressible Stream

Consider the system schematically represented in Fig. 1. A gas flows into a constant-area duct at an initial velocity V_1 and Mach number M_1. From section 1 to the final section 2, liquid particles are added to the flow. Solid particles would have the same effect; the name spray used throughout the paper is to be understood to include a dispersion of solid particles as well. Section 2 is sufficiently far downstream so that both phases have reached the same velocity V_2.

Now, the addition of particles to the gas alters the sound speed because the particles add to the inertia of the two-phase mixture but leave its compressibility untouched, if the liquid volume fraction is much smaller than that of the gas. For this limit, and assuming that the particles are small enough to instantly follow the velocity variations of the gas flow once they move at the same average speed, Wallis[1] gives the following expression for the speed of sound of the two-phase mixture:

$$c_2 = \sqrt{kR_gT_2/(1 + ms)} \tag{1}$$

where R_g is the ideal gas constant of the gas without particles, ms is the particle mass flow rate divided by the gas mass flow rate, and k is the specific heat ratio of the gas. A more detailed analysis by Chenoweth and Paolucci[3] leads to the same result for the limit of negligible particle-volume fraction and no thermal interaction between gas and particles. Equation (1) is based on the assumption that the only effect of the particles is to increment the inertia of the mixture.

When the injecting spray velocity V_L is different from V_1, there will be momentum transfer between the two phases. In a real situation this would be accompanied by some mass and heat transfer as well. The rate at which momentum transfer occurs, relative to those of heat and momentum transfer, would be controlled by the Prandtl and Lewis numbers of the gas, in addition to the Biot numbers of the particles. The problem could become quite difficult if we try to cover all of the cases, some having significant heat and mass transfer because of good diffusivity in both phases and some others just the opposite if one of the phases has a poor diffusivity.

To simplify the problem down to its basics, therefore, the assumption will be made that there is no mass nor heat transfer between the phases. This would correspond to the case where the gas has a high Prandtl number or the particles have a low internal diffusivity, which is close to true in many cases. In addition, the assumption will be made that there is no heat nor momentum transfer with the

Received 26 June 1998; revision received 14 May 1999; accepted for publication 7 June 1999.

*Associate Professor, Department of Mechanical, Materials, and Aerospace Engineering, 10 W. 32nd Street.

duct walls. This will allow us to separate the effect of the particles from wall effects that have been studied already long ago.

With these assumptions the final pressure, velocity, and temperature must be determined by the conservation laws over a control volume extending between sections 1 and 2, that is,

Continuity in the gas phase:

$$\rho_1 V_1 = \rho_2 V_2 \tag{2}$$

Momentum, with friction allowed only between phases:

$$(P_1 - P_2)A = \overset{\circ}{m}_L(V_2 - V_L) + \rho_1 A V_1 (V_2 - V_1) \tag{3}$$

Energy, assuming no heat transfer between phases or with the walls:

$$\rho_1 A V_1 \left(CpT_2 + \tfrac{1}{2}V_2^2 - CpT_1 - \tfrac{1}{2}V_1^2\right) - \overset{\circ}{m}_L\left(\tfrac{1}{2}V_L^2 - \tfrac{1}{2}V_2^2\right) = 0 \tag{4}$$

Equations (2–4) can be put in dimensionless terms using the Mach numbers M_L and M_1 and the relative temperature τ, defined by

$$M_L = \frac{V_L}{c_1} = \frac{V_L}{\sqrt{kR_g T_1}} \tag{5}$$

$$M_2 = \frac{V_2}{c_2} = \frac{V_2}{\sqrt{kR_g T_2/(1 + ms)}} \tag{6}$$

$$\tau = T_2/T_1 \tag{7}$$

These equations, in dimensionless terms, and substituting continuity and the ideal gas equation into the other equations, become

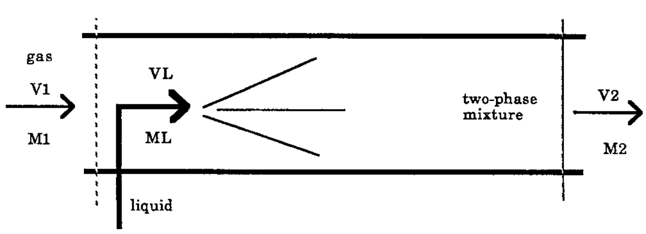

Fig. 1 Schematic of the flow and parameters.

Momentum:

$$\tau(kM_2 + 1/M_2)^2(1 + ms) = (kM_L ms + kM_1 + 1/M_1)^2 \tag{8}$$

Energy (adiabatic):

$$\tau\left[2/(k-1) + M_2^2\right] = 2/(k-1) + M_1^2 + M_L^2 ms \tag{9}$$

An easy way to see how the particles affect the flow is to plot the temperature, pressure, velocity, and Mach number M_2 obtained from these equations as a function of the mass fraction ms for a constant value of the injection Mach number M_L. If the particles are gradually released along the duct, the actual variation of these quantities along the duct will be very similar to these plots. Figure 2 was obtained for $M_1 = 0.7$ and a smaller value of $M_L = 0.6$. When M_L is smaller than M_1, the flow velocity rises as ms is increased, and at the same time the pressure decreases. The flow Mach number rises; when it reaches unity, it cannot rise any further because the equations do not give real solutions for values of ms greater than those that give $M_2 = 1$. This effect is quite similar to that of flow with friction or heat addition. The same behavior is obtained whether the initial flow is subsonic or supersonic.

When M_L is larger than M_1, on the contrary, the pressure rises and the flow velocity decreases. Figure 3 shows that after an initial rise of the Mach number it starts to drop again, and it never reaches unity. Except for the initial rise, this behavior is similar to that of a flow with cooling.

Equations (8) and (9) are of second degree in ms, M_L and M_2 (M_1 is a fixed parameter, in all that follows). For each value of ms and M_2, there will be two or none real values of M_L. The same can be said of ms, given the values of M_L and M_2.

There are some combinations of M_L and ms for which there is no real solution; these combinations correspond to choking conditions. To determine these conditions, let us first determine what combinations of these parameters will lead to a final sonic state ($M_2 = 1$). Substituting this into Eqs. (8) and (9), and eliminating τ, the following relationship is obtained:

$$(1 + k)\left[2 + (k-1)\left(M_1^2 + M_L^2 ms\right)\right](1 + ms) = (kM_L ms + kM_1 + 1/M_1)^2 \tag{10}$$

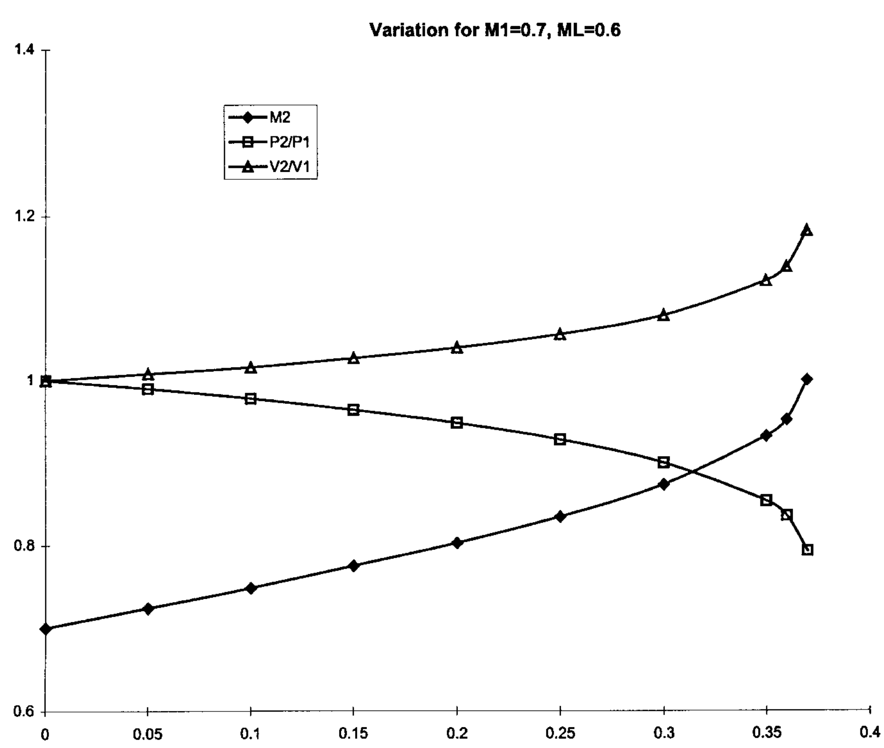

Fig. 2 Variation of several flow parameters vs ms, when $M_L < M_1$.

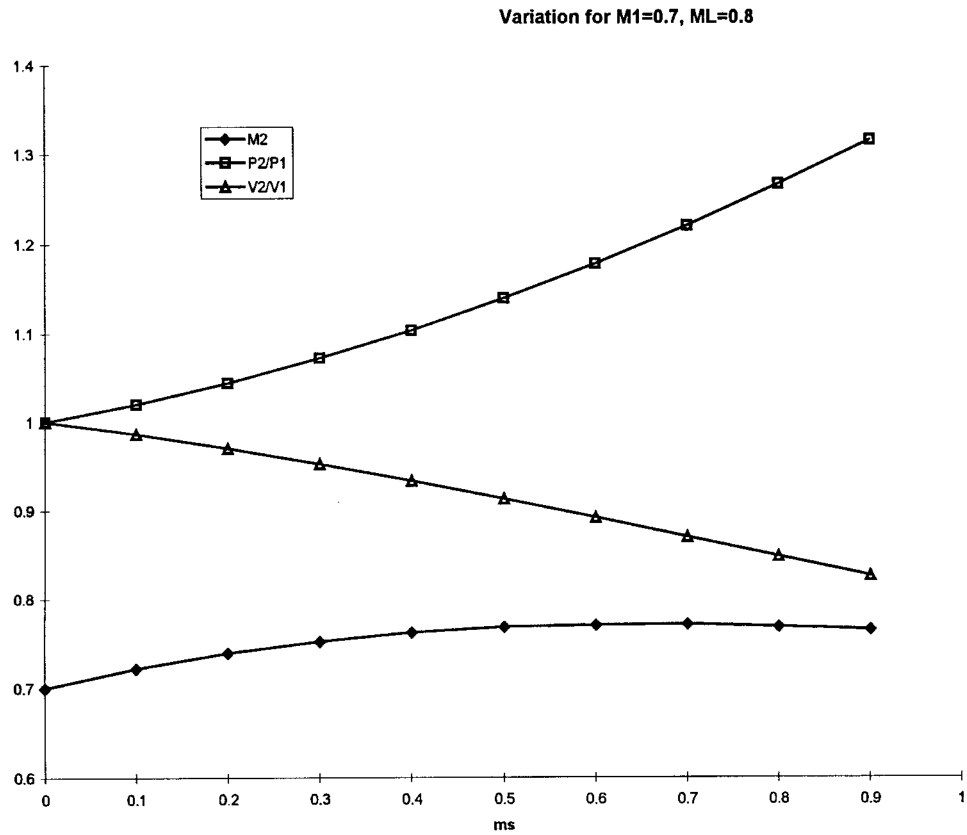

Fig. 3 Variation of several flow parameters vs *ms*, when $M_L > M_1$.

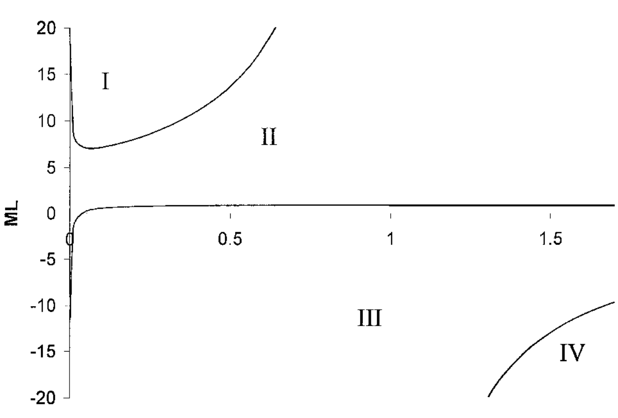

Fig. 4 Full plot of the $M_2 = 1$ choking line, showing the solution regions in the *ms*-M_L plane.

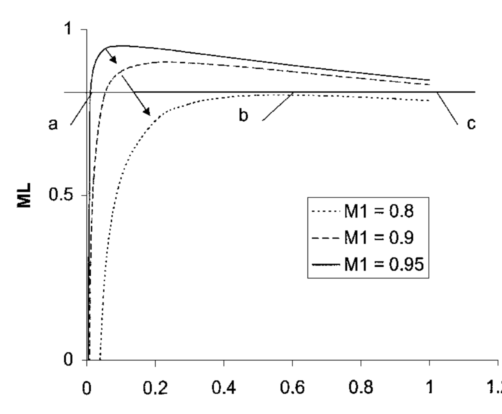

Fig. 5 Change in the $M_2 = 1$ choking line as *ms* is increased beyond choking conditions for $M_L < M_1$.

Equation (10) is plotted in the M_L-ms plane in Fig. 4 for $M_1 = 0.8$ and physically meaningful (positive) values of ms. There is no real solution when the injection Mach number M_L lies between 0.8 (that is, M_1) and 7.0, in the hypersonic region. It seems paradoxical because it is possible to reach sonic conditions by injecting particles at a velocity V_1 or less because there is a solution for $M_L = M_1$ and $ms = 1/M_1^2 - 1$, on the local maximum of the curve between regions II and III in Fig. 4 (this local maximum is not visible due to the scale, but it can be seen in Figs. 5 and 8). Sonic conditions cannot be reached by injecting particles at a higher speed (which would seemingly tend to increase the final velocity V_2), nor at a higher rate (which would tend to lower the speed of sound to an even lower value).

This paradox can be better understood by replacing M_2 back into the equations. Given certain values of ms and M_L, the final Mach number M_2 is given by the following equation [obtained from Eqs. (8) and (9), by eliminating τ]:

$$\frac{(kM_2 + 1/M_2)^2}{2/(k-1) + M_2^2} = \frac{(kM_L ms + kM_1 + 1/M_1)^2}{(1+ms)\left[2/(k-1) + M_1^2 + M_L^2 ms\right]} \qquad (11)$$

Equation (11) is of fourth degree in M_2; for each pair of values of ms and M_L, there are either two real values of M_2 (one subsonic and one supersonic) or none. The other two solutions are always complex. Regions II and IV in Fig. 4 allow two real values of M_2, whereas regions I and III do not allow any real solution. The two solutions in regions II and IV, M_2 and $M_{2'}$, are related by this equation:

$$\frac{(kM_2 + 1/M_2)^2}{1 + [(k-1)/2]M_2^2} = \frac{(kM_{2'} + 1/M_{2'})^2}{1 + [(k-1)/2]M_{2'}^2} \qquad (12)$$

which yields two solutions: the trivial one

$$M_{2'} = M_2$$

and

$$M_{2'} = \sqrt{\frac{M_2^2 + 2/(k-1)}{[2k/(k-1)]M_2^2 - 1}} \tag{13}$$

This is the same relationship that binds the changing values of the Mach number across a normal shock wave. One solution is subsonic, and the other is supersonic. Normally, the only physically possible solution is the subsonic one because the supersonic solution would involve a spontaneous decrease in entropy, against the second law of thermodynamics, as we shall see later.

In regions I and III of Fig. 4, there is no solution for M_2 from Eq. (11). Because region III cannot be reached, neither will region IV be attainable, leaving only region II for physical solutions. Because ms and M_L are both independently set parameters, which could be varied at will in an experiment, there should be at least one solution for M_2 if there is any flow at all. The problem is similar to that of arbitrary heat addition or wall friction: a point is reached in which any further variation in a certain direction does not change the Mach number, which continues at its sonic value, but rather causes the initial conditions to be changed. In the case of this study, any attempt to inject particles with a combination of the parameters ms and M_L within regions I and III will result in a choked exit flow, with $M_2 = 1$, and a decrease of M_1 below the original value if M_1 was subsonic. If the flow was originally supersonic, it will result in the upstream propagation of a shock wave that brings M_1 to a subsonic value.

It is interesting to see how the choking state will evolve as ms is increased from zero—that is, as particles are added—at a constant subsonic $M_L < M_1$. First M_2 will increase with ms, until $M_2 = 1$ is reached (for instance, point a in Fig. 5). A further increase of ms will maintain choked flow, but M_1 will be decreased. Physically, this will be caused by a sonic compression wave traveling upstream. The change in M_1 will continue until the higher choking point is reached (b in Fig. 5), which is on the maximum of the curve of $M_2 = 1$, for a smaller value of $M_1 = M_L$. A further increase of ms will maintain $M_1 = M_L$, but the final state will be supersonic and the phase interaction will be ideally frictionless (point c in Fig. 5 is an example).

The downstream pressure will be adjusted by an expansion wave or a system of oblique shock waves at the end of the duct, like in a supersonic nozzle. If there is sufficient backpressure for a normal shock, the shock will merge with the two-phase interaction, so that a sonic flow will never in effect be reached.

If M_1 is subsonic and $M_L > M_1$, the exit Mach number will never reach unity: it will start decreasing as ms increases, after a short initial increase. The flow velocity will decrease continuously as in a divergent duct. If M_1 is supersonic and $M_L > M_{1'}$ [conjugate of M_1 across a normal shock, as in Eq. (13)], the flow will stay supersonic, as long as the downstream pressure is low enough to prevent the formation of a normal shock wave.

Entropy Variation in Adiabatic Spray Mixing

Another way to look at the interaction is to consider the change in the entropy of the gas (recall that the thermodynamic state of the particles does not change in this analysis) before and after it interacts with the spray and both acquire the same velocity. The variation of specific entropy of the gas from 1 to 2, after substituting the appropriate equations, is given by

$$\frac{S_2 - S_1}{R_g} = \frac{k+1}{2(k-1)} \ln\frac{T_2}{T_1} - \ln\frac{M_1}{M_2} - \frac{1}{2}\ln(1+ms) \tag{14}$$

This equation is obtained directly from the temperature and pressure change of the gas over the complete control volume. Following a constant M_2 line, the entropy variation looks as shown in Fig. 6. For a subsonic M_2 the value of the entropy change has a zero and a local minimum at $M_L = M_1$, the higher choking point (b in Fig. 5). The conjugated supersonic Mach number $M_{2'}$ (having the same line in the M_L-ms plane) gives a local entropy maximum at the crossing with $M_L = M_1$, this time on the right side of the higher choking point. Both curves are separated by a constant entropy offset of value:

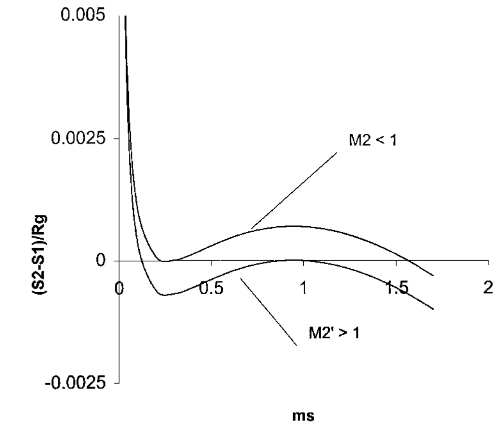

Fig. 6 Entropy variations at constant M_2 for $M_2 = 0.9$, and its supersonic conjugate $M_{2'} = 1.115$.

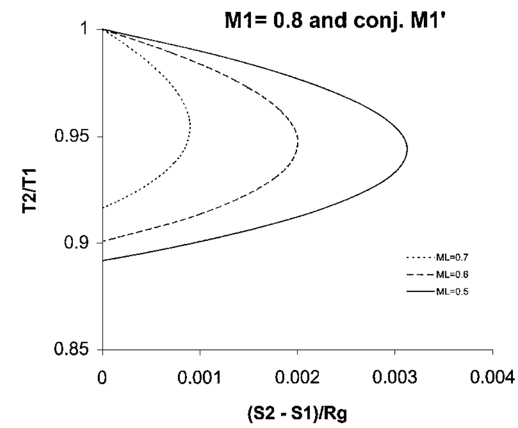

Fig. 7 Temperature-entropy diagram for the two-phase interaction at constant $M_1 = 0.8$ and three different initial particle Mach numbers.

$$\frac{\Delta S_{\text{sub}} - \Delta S_{\text{super}}}{R_g} = \frac{k}{k-1} \ln\left\{\frac{M_{2'}}{M_2}\sqrt{\frac{1+[(k-1)/2]M_2^2}{1+[(k-1)/2]M_{2'}^2}}\left(\frac{1+kM_2^2}{1+kM_{2'}^2}\right)\right\} - \ln\left(\frac{1+kM_2^2}{1+kM_{2'}^2}\right) \tag{15}$$

which is the same as the entropy generated by a normal shock wave that lowers the Mach number from an initial $M_{2'} > 1$, to its conjugated $M_2 < 1$, given by Eq. (13). It appears, then, that the solutions in the subsonic M_2 domain are those in the supersonic M_2 domain, followed by a normal shock wave. This makes sense, because the assumptions made for the phase interaction process (adiabatic, conservation of mass and momentum) would yield a normal shock wave in the absence of the particles. The subsonic solutions can be thought of as the superposition of a supersonic solution and a normal shock wave, with both processes (mixing and entropy production) occurring simultaneously, so that at no time the entropy decreases.

One can see in Fig. 6 that a subsonic M_2 normally corresponds to a positive ΔS, whereas a supersonic $M_{2'}$ solution normally gives a negative ΔS and is, therefore, impossible by the second law of thermodynamics for an adiabatic process. When $M_L = M_1$, the interaction is frictionless, and we obtain, from Eqs. (8) and (9):

$$M_2 = M_1(1+ms)^{\frac{1}{2}} \tag{16}$$

$$T_2 = T_1 \tag{17}$$

and, therefore,

$$\Delta S/R_g = 0 \tag{18}$$

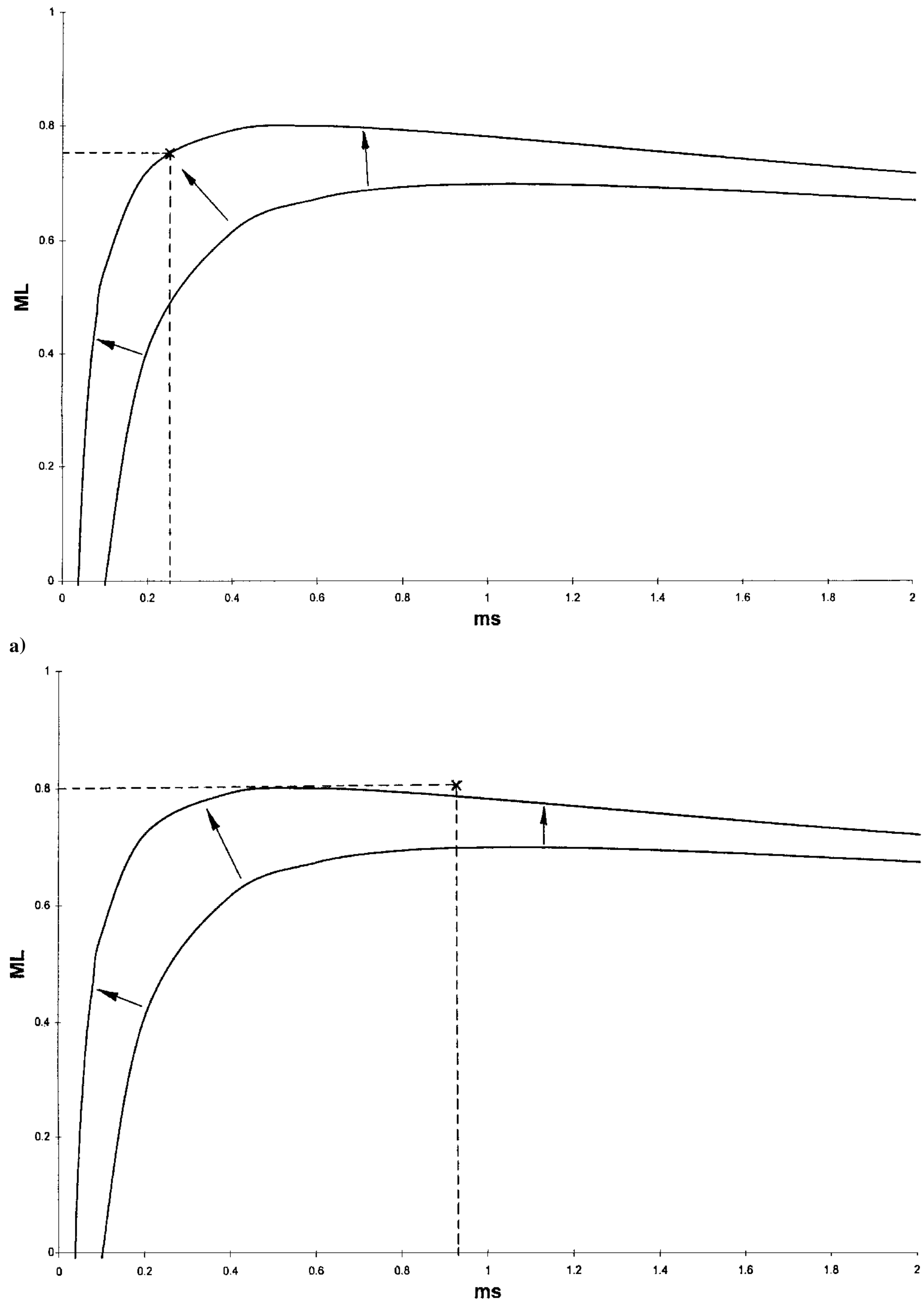

Fig. 8 Change in the choking line and flow condition as the downstream pressure is decreased: *ms* a) smaller or b) greater than that at the maximum.

Thus, the flow will be able to pass from subsonic to supersonic by adding spray particles, only if these are injected at the same speed as that of the original gas flow.

Many supersonic M_2 cases would yield $\Delta S < 0$, in violation of the second law of thermodynamics. If the second law is to be preserved, then ΔS will have to be greater or equal than zero: when $M_1 < 1$, an $M_2 > 1$ can only be possible on the line $\Delta S = 0$. For this to occur, given that M_L and *ms* are arbitrarily chosen and the downstream pressure can be lowered to whatever value is required, the upstream conditions must change, as in the cases of choking. In other words, M_1 will remain locked to a value equal to M_L.

The temperature-entropy locus $(\tau, \Delta S/R_g)$ of the final state of the interaction, as the relative particle flow *ms* is increased from zero, is plotted in Fig. 7, for $M_1 = 0.8$ and its conjugate $M_{1'}$, and constant $M_L = 0.5, 0.6$, and 0.7. The exit Mach number M_2 is subsonic in the upper branches of the curves and supersonic in the lower branches. In the upper branch as *ms* increases, τ decreases and ΔS increases, up to a maximum value of ΔS. In the lower branch as *ms* increases, τ increases and ΔS increases, up to a maximum ΔS. The analogy with the well-known Fanno line of flow with friction and the Rayleigh line of flow with heat addition is apparent.

The choking condition corresponds to the maximum of the entropy generation. Unlike in those classic cases, however, choking is not always reached as *ms* increases. For a subsonic M_L slightly larger than M_1, the flow will always remain subsonic, and M_2 will reach a maximum for a certain value of *ms* and will decrease as *ms* increases beyond this value, with the entropy increasing all of the time. Likewise, the curve corresponding to the supersonic conjugate $M_{1'}$ always stays in the supersonic region. When M_L is larger than the subsonic M_1, the subsonic and supersonic branches of the temperature-entropy locus do not meet, as the entropy increases. This is what would be obtained from the classical equations of flow with friction, if a negative friction coefficient is used. Only for $M_L = M_1$, the flow will be able to pass from subsonic to supersonic as *ms* increases, because the interaction would then be isentropic, provided there are no other sources of irreversibility.

Effect of the Downstream Pressure

Which solution for M_2 (subsonic or supersonic) appears in a real case will be determined by the downstream pressure. If this pressure is high enough to support a normal shock wave, then the exit flow will be subsonic. If, on the contrary, the downstream pressure is low, then a normal shock cannot be supported, and M_2 may be larger than unity. Between these, other cases can appear in which the particle-gas interaction is followed by an oblique shock wave.

In the subsonic M_2 cases a variation in the exit pressure will automatically induce a change in M_1 whether M_L is smaller than M_1 or not. Starting from a low value of M_1, when the downstream pressure is decreased, the initial Mach M_1 will increase. For a low value of *ms*, the left side of the choking line will eventually meet the actual M_L-*ms* point. This is illustrated in Fig. 8a, which plots the change in the choking line $M_2 = 1$ as M_1 increases. At this point the flow will be choked, and any further decrease in pressure will not be transmitted upstream, but rather will produce a set of Prandtl–Meyer expansion waves (for a two-dimensional duct) or their three-dimensional equivalent, beginning at $M = 1$.

If, on the contrary, *ms* is large enough so that the peak of the choking line would eventually be placed at a lower value of *ms* with $M_1 = M_L$ (see Fig. 8b), then the process will be totally different: when M_1 has reached the value of M_L, the flow will be able to become supersonic in an isentropic way. Physically, this transition can be viewed as the separation of the shock component of the two-phase interaction, which will then be decomposed into an isentropic particle dispersion process, followed by a normal shock wave. If the exit pressure falls below the value required for a normal shock, the isentropic interaction component with $M_1 = M_L$ will be maintained, but the shock component will move down the duct until it is transformed into a system of oblique shock waves at the duct exit, which will adapt the duct pressure after the interaction to the downstream pressure. A further downstream pressure drop will cause the oblique shock waves to weaken and disappear and to be replaced by a set of expansion waves.

In the supersonic M_1 and M_2 cases the two-phase interaction will be fully supersonic, followed by oblique shock or expansion waves. If the downstream pressure reaches the level that would produce a normal shock wave, this shock wave will propagate upstream until the whole flow is made subsonic. A further increase in the downstream pressure will lower M_1, accordingly.

Conclusions

1) There is an analogy between certain cases of interaction of a spray with a compressible gas flow, and heat addition into a gas flow, or flow with friction: Mach number changes leading to eventual choking are predicted.

2) Unlike in the case of heat addition or friction, a point can be reached where the interaction would be isentropic, leading to a continuous change of the Mach number from subsonic to supersonic.

3) When the particles are injected at a speed moderately greater than that of the flow, choking is not possible, and the flow cannot go from subsonic to supersonic, and vice versa.

4) The temperature-entropy locus for this interaction is composed of series of lines that have the choking condition as their limit, for initially subsonic and initially supersonic flows, as long as the particles are injected at a smaller speed than the gas flow.

References

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